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LETTER TO THE EDITOR

On parasupersymmetry and remarkable Lie structures

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Abstract. On the basis of recent physical results in parasupersymmetric quantum mechanics, we construct new Lie structures that we call Lie parasuperalgebras. They are characterised by *double commutation* relations between even *and* odd generators. Parasuperalgebras do contain superalgebras in correspondence with the physical inclusion of supersymmetry in parasupersymmetry.

Supersymmetry (Fayet and Ferrara 1977, 1985) in particle physics has involved the main motivations for the development(s) of Lie superalgebras (or Lie graded algebras) as a recent field in mathematics (Kac 1977). Due to the simultaneous description of bosons and fermions and associated creation and annihilation operators, supersymmetry led to a generalised Lie product including the usual Lie bracket $[\cdot,\cdot]$—the so-called commutator—*and* the new bracket $\{\cdot,\cdot\}$—the so-called anticommutator—in such a way that *even* ($\mathscr{E}$) bosonic operators and *odd* ($\mathscr{O}$) fermionic operators do generate a closed Lie superalgebra according to the 'structure relations':

$$[\mathscr{E},\mathscr{E}]\sim\mathscr{E} \qquad [\mathscr{E},\mathscr{O}]\sim\mathscr{O} \qquad \{\mathscr{O},\mathscr{O}\}\sim\mathscr{E}. \tag{1}$$

In quantum mechanics, supersymmetry was originally developed by Witten (1981). If we restrict ourselves, for example, to the so-called $(N=2)$-supersymmetric quantum mechanics, we immediately deal with the superalgebra generated by the even super-Hamiltonian H and the two odd supercharges Q^1, Q^2 or Q, $Q^\dagger$. The latter are such that:

$$[H,Q]=0 \qquad [H,Q^\dagger]=0 \tag{2a}$$

$$\{Q,Q^\dagger\}=H \qquad \{Q,Q\}=0 \qquad \{Q^\dagger,Q^\dagger\}=0 \tag{2b}$$

leading to the superalgebra called sqm(2) (Witten 1981).

Parasupersymmetry (Rubakov and Spiridonov 1988, Beckers and Debergh 1990) has very recently superposed bosons and parafermions (or parabosons and parafermions). In particular, we can also speak here about $(N=2)$-parasupersymmetric quantum mechanics for describing oscillator-like interactions between bosons and parafermions with remarkable *physical* properties. Let us just mention a complete determination of the corresponding parasuperspectrum containing a null energy value associated with a unique fundamental ground state and leading to *exact* parasupersymmetries (Beckers and Debergh 1990). Such a parasupersymmetric theory is subtended by an even parasuper-Hamiltonian $H_{\rm PSS}$ and two odd parasupercharges characterised by the following structure relations:

$$[H_{\rm PSS},Q]=0 \qquad [H_{\rm PSS},Q^\dagger]=0 \qquad Q^3=0 \qquad Q^{\dagger 3}=0 \tag{3a}$$

and

$$[Q,[Q^\dagger, Q]] = 2QH_{\rm PSS} \qquad [Q^\dagger,[Q, Q^\dagger]] = 2Q^\dagger H_{\rm PSS}. \tag{3b}$$

Let us point out that these relations (3) are *not* equivalent to those originally determined by Rubakov and Spiridonov (1988), a result already discussed by us (1990).

We propose now to put these relations (3) in correspondence with a new generalised Lie bracket leading to the definition of a new Lie structure: the Lie parasuperalgebra Psqm(2). In a parallel way with the correspondence (1)↔(2), we effectively put in evidence *necessary double commutators* as the new product characterising the relations (3). We ask then for *six* types of double commutation relations between even and odd operators, as follows:

$$[\mathscr{E},[\mathscr{E},\mathscr{E}]]\sim[\mathscr{E},\mathscr{E}] \tag{4a}$$

$$[\mathscr{E},[\mathscr{E},\mathscr{O}]]\sim[\mathscr{E},\mathscr{O}] \tag{4b}$$

$$[\mathscr{O},[\mathscr{E},\mathscr{E}]]\sim[\mathscr{O},\mathscr{E}] \tag{4c}$$

$$[\mathscr{O},[\mathscr{O},\mathscr{O}]]\sim[\mathscr{O},\mathscr{E}] \tag{5a}$$

$$[\mathscr{O},[\mathscr{E},\mathscr{O}]]\sim[\mathscr{O},\mathscr{O}] \tag{5b}$$

$$[\mathscr{E},[\mathscr{O},\mathscr{O}]]\sim[\mathscr{E},\mathscr{E}]. \tag{5c}$$

We notice that all the relations (4) only contain usual admissible Lie commutators—see (1)—between even-even and even-odd operators while the other three relations (5) contain new commutators between odd-odd operators. We also point out that the double commutators in the relations (4) and (5) respectively correspond to each other through the simultaneous substitution $\mathscr{E}\to\mathscr{O}$ *and* $\mathscr{O}\to\mathscr{E}$. Finally we remark that, amongst the six relations (4) and (5), only (5*a*) is, on the one hand, really disturbing *at the start* as a double commutator between all *odd* operators but is, on the other hand, precisely the only one which *is* convenient in connection with our relations (3*b*). In particular, it is trivial to show that our context associated with the relations (3) is evidently absorbed by the structure relations (4) and (5).

Let us now give a general form to these double commutators in the following way. By denoting $\{K_a, (a=0,1,\ldots,k), K_0\equiv I\}$ the set of $(k+1)$ *even* generators where I is the identity operator and $\{L_\alpha, (\alpha=1,2,\ldots,l)\}$ the set of l *odd* generators, we get in correspondence with (4) and (5)

$$[K_a,[K_b,K_c]] = A^{od}_{abc}K_oK_d \tag{6a}$$

$$[K_a,[K_b,L_\alpha]] = B^{\alpha\beta}_{ab\alpha}K_oL_\beta \tag{6b}$$

$$[L_\alpha,[K_a,K_b]] = C^{o\beta}_{\alpha ab}K_oL_\beta \tag{6c}$$

$$[L_\alpha,[L_\beta,L_\gamma]] = D^{\delta a}_{\alpha\beta\gamma}L_\delta K_a \tag{6d}$$

$$[L_\alpha,[K_a,L_\beta]] = E^{bc}_{\alpha a\beta}K_bK_c \tag{6e}$$

$$[K_a,[L_\alpha,L_\beta]] = F^{bc}_{a\alpha\beta}K_bK_c \tag{6f}$$

where the sets of A-, B-, C-, D-, E-, F-coefficients play the role of new structure constants verifying in particular the following Jacobi relations:

$$\begin{aligned} &A^{od}_{abc}+A^{od}_{cab}+A^{od}_{bca}=0 \qquad && B^{o\beta}_{ab\alpha}-B^{o\beta}_{ba\alpha}=C^{o\beta}_{\alpha ba} \\ &D^{\delta a}_{\alpha\beta\gamma}+D^{\delta a}_{\gamma\alpha\beta}+D^{\delta a}_{\beta\gamma\alpha}=0 \qquad && E^{bc}_{\alpha a\beta}-E^{bc}_{\beta a\alpha}-F^{bc}_{a\alpha\beta}. \end{aligned} \tag{7}$$

In our particular case we have $k=1$ and $l=2$ and the set of structure constants lead to a parasuperalgebra that we call Psqm(2) generated essentially by H_{PSS}, Q and $Q^{\dagger}$. Let us justify this appellation in connection with the Witten superalgebra sqm(2) ≡ (2). In fact, it is easy to show that equations (3*b*) can be respectively written as

$$\{Q, \{Q^{\dagger}, Q\} - H_{\mathrm{PSS}}\} = \{Q^{\dagger}, \{Q, Q\}\} \tag{8a}$$

and

$$\{Q^{\dagger}, \{Q^{\dagger}, Q\} - H_{\mathrm{PSS}}\} = \{Q, \{Q^{\dagger}, Q^{\dagger}\}\}. \tag{8b}$$

We immediately conclude by (2) and (3) that

$$\mathrm{sqm}(2) \Rightarrow \mathrm{Psqm}(2) \tag{9}$$

but that the converse is not true. The inclusion of supersymmetric quantum mechanics is thus ensured in our developments and we notice, in particular, that double commutators can be simply related to anticommutators, the last ones being absolutely necessary for graded structures.

The first non-trivial context for parafermions (Ohnuki and Kamefuchi 1982) being associated with the order $p=2$ and with a $D^{(p/2)} = D^{(1)}$ representation of the compact algebra so(3), we know that parafermionic operators are represented by 3×3 matrices and that the corresponding parasuper-Hamiltonian takes the form (Beckers and Debergh 1990)

$$H_{\mathrm{PSS}} = \frac{1}{2}(p^2 + \omega^2 x^2)\mathbb{1}_3 + \frac{\omega}{2}\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = H_{\mathrm{b}} + H_{\mathrm{pf}} \tag{10}$$

obtained from the parasupercharges

$$Q\begin{pmatrix} 0 & 0 & 0 \\ p+\mathrm{i}\omega x & 0 & 0 \\ 0 & p-\mathrm{i}\omega x & 0 \end{pmatrix} \qquad Q^{\dagger} = \begin{pmatrix} 0 & p-\mathrm{i}\omega x & 0 \\ 0 & 0 & p+\mathrm{i}\omega x \\ 0 & 0 & 0 \end{pmatrix} \tag{11}$$

H_{b} and H_{pf} being the physical bosonic and parafermionic Hamiltonians. The inclusion PSS ⊃ SS here is trivially recognised by suppressing the first row and column in the above formulation. This also leads to a supplementary possibility to define new odd operators in this context. Let us exploit such a freedom and define the two new parasupercharges

$$P = \begin{pmatrix} 0 & 0 & 0 \\ -(p+\mathrm{i}\omega x) & 0 & 0 \\ 0 & p-\mathrm{i}\omega x & 0 \end{pmatrix} \qquad P^{\dagger} = \begin{pmatrix} 0 & -(p-\mathrm{i}\omega x) & 0 \\ 0 & 0 & p+\mathrm{i}\omega x \\ 0 & 0 & 0 \end{pmatrix}. \tag{12}$$

It is easy to show that we have

$$[H_{\mathrm{PSS}}, P] = 0 \qquad [H_{\mathrm{PSS}}, P^{\dagger}] = 0 \qquad P^3 = 0 \qquad P^{\dagger 3} = 0 \tag{13a}$$

$$[P, [P^{\dagger}, P]] = 2PH_{\mathrm{PSS}} \qquad [P^{\dagger}, [P, P^{\dagger}]] = 2P^{\dagger}H_{\mathrm{PSS}}. \tag{13b}$$

This is always possible and we are thus going to define a larger parasuperalgebra ($k=1$ but $l=4$) called Psqm(4). Its typical structure relations evidently include the equations

(3) and (13) but also all the supplementary non-trivial ones issued from equation (5*a*), i.e.

$$
\begin{aligned}
&[P,[P^\dagger, Q^\dagger]] = 4Q^\dagger H && [P^\dagger,[P,Q]] = 4QH \\
&[Q^\dagger,[P,P^\dagger]] = 2Q^\dagger H && [Q,[P^\dagger,P]] = 2QH \\
&[P^\dagger,[Q^\dagger,P]] = -6Q^\dagger H && [P,[Q,P^\dagger]] = -6QH \\
&[P,[Q,Q^\dagger]] = -2PH && [P^\dagger,[Q,Q^\dagger]] = 2P^\dagger H \\
&[Q^\dagger,[P,Q]] = -4PH && [Q,[Q^\dagger,P^\dagger]] = 4P^\dagger H \\
&[Q,[Q^\dagger,P]] = 6PH && [Q^\dagger,[P^\dagger,Q]] = -6P^\dagger H \\
&[Q,[P^\dagger,Q]] = 6PH && [Q^\dagger,[P,Q^\dagger]] = 6P^\dagger H \\
&[P,[Q^\dagger,P]] = 6QH && [P^\dagger,[Q,P^\dagger]] = 6Q^\dagger H.
\end{aligned}
\tag{14}
$$

Through the inclusion PSS $\supset$ SS, we immediately see that the parasupercharges Q and P ($Q^\dagger$ and $P^\dagger$) reduce to the only one Q ($Q^\dagger$) in the supersymmetric context so that

$$\mathrm{Psqm}(4) \supset \mathrm{Psqm}(2) \supset \mathrm{sqm}(2) \tag{15}$$

as expected.

The above results mainly point out the inclusion of Lie superalgebra(s) *inside* our larger structures that we call Lie parasuperalgebra(s) for evident reasons.

Here is not the right place to develop further the study of the structure relations (6) and (7). We claim and have shown that such parasuperalgebras exist and are more general structures than superalgebras. This letter is also a call to mathematicians for giving information and properties on the above proposed Lie structures. We obtain here, perhaps, a new example of interaction between physics and mathematics.

Let us end this letter with a few remarks. First it is evident that, for arbitrary N, N-supersymmetric quantum mechanics can be extended to the N-parasupersymmetric context and, consequently, that we easily can define the parasuperalgebra Psqm(N) such that

$$\mathrm{Psm}(N) \supset \mathrm{sqm}(N). \tag{16}$$

Secondly, let us mention that *all* the structure relations (3) are included in the formulation (6): it is in fact possible to extract $Q^3 = 0 = Q^{\dagger 3}$ from *double* commutation relations as Ohnuki and Kamefuchi (1982) have shown, in particular, for parafermionic creation and annihilation operators. Finally let us notice that, in terms of the Hermitian supercharges

$$Q_1 = \frac{1}{2}(Q + Q^\dagger) \qquad Q_2 = \frac{\mathrm{i}}{2}(Q - Q^\dagger) \tag{17}$$

the typical structure relations of Psqm(2) can once again be expressed in terms of double commutators. We have, for example,

$$[Q_1,[Q_1,Q_2]] = Q_2 H_{\mathrm{PSS}} \qquad [Q_2,[Q_2,Q_1]] = Q_1 H_{\mathrm{PSS}} \tag{18}$$

in correspondence with (5*a*) or (6*d*). These relations can once again be expressed in terms of anticommutators as follows:

$$\{Q_2, \{Q_1, Q_1\} - \tfrac{1}{2} H_{\mathrm{PSS}}\} = \{Q_1, \{Q_1, Q_2\}\} \tag{19a}$$

and

$$\{Q_1, \{Q_2, Q_2\} - \tfrac{1}{2}H_{PSS}\} = \{Q_2, \{Q_1, Q_2\}\} \tag{19b}$$

showing the immediate inclusion of sqm(2) expressed in terms of the supercharges Q_1 and Q_2 (Witten 1981).

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